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Lift Order Problems for Ordinary Differential Equations on Manifolds

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For $k \geq 0$, let π_k: $T^{k+1}(M) = T(T^k(M)) \rightarrow T^k(M)$ denote the $(k + 1)$th iterated tangent bundle in relation to a base manifold $T^0(M) = M$. Let V represent a possibly nonstationary vector field over $T^k(M)$, and let Q be a subset/submanifold in $T^k(M)$. Sufficient conditions (and, when V is completely integrable in Q, necessary and sufficient conditions) are established to ensure that all solutions g to $y' = V(t, y)$ lying entirely in Q have the form $G = f^{[k]}$, where $f^{[k]}$ is the kth-order differential lift of a curve f lying in M. The relevance of the issue for higher order dynamical systems (especially in mechanics) is discussed. Higher order involutions and complete vector field lifts are examined from the viewpoint of the differential equations they present. Collateral results on the general solvability of initial value problems are obtained and numerous examples are discussed in detail.

1. INTRODUCTION

The underlying scalar field is fixed: $K = \mathrm{R}$ or C. Let M be a differentiable manifold modeled on a K-Banach space F. Let $(T^k(M), \pi_k)_{k\geq 0}$ represent the full tangential resolution for M, i.e., $T^0(M) = M$, and π_k: $T^{k+1}(M) = T(T^k(M)) \rightarrow T^k(M)$ is the standard tangent bundle map.

With U open in K, let f: $U \rightarrow M$ be a differentiable K-curve in M. Then f': $U \rightarrow T(M)$ will denote the standard differential lift of f to $T(M)$. Iteratively, with $f^{[0]} = f$, let $f^{[k+1]}$: $U \rightarrow T^{k+1}(M)$ be given by $f^{[k+1]} = (f^{[k]})'$.

Let V be a K-dependent $(k + 1)$-level differentiable vector field over M. That is, with W open in K, V: $W \times T^k(M) \rightarrow T^{k+1}(M)$ is differentiable and $(\pi_k \circ V)(t, y) = y$ always holds. By the *differential lift equation (= ODE) associated with* V we shall mean the formal expression $y' = V(t, y)$, a *solution*

To the memory of my teacher and friend M. Kuga (1928–1990).

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to which is a differentiable K-curve g: $U \rightarrow T^k(M)$, with U open in W, such that $g' = V(t, g(t))$ always holds. By an *initial value problem associated with V* we mean an expression (V, s, p), where s and p are chosen from W and $T^k(M)$, respectively, a *solution* to which is a solution g to $y' = V(t, y)$ such that $g(s) = p$. [We shall regard the case of a *stationary* vector field V: $T^k(M) \rightarrow T^{k+1}(M)$ as a variant of the foregoing idea, the relevant equation being $y' = V(y)$, a solution to which is a K-curve g such that $g'(t) = V(g(t))$ always holds.]

Let Q be any nonempty subset of $T^k(M)$. With V as above, let ord(V, Q) denote the maximum integer $j + 1 \leq k + 1$ such that *every* g solving $y' = V(t, y)$ and lying entirely in Q can be expressed $g = f^{[j]}$, where f is a K-curve taking its values in $T^{k-j}(M)$. We call ord(V, Q) the *order of V relative to Q*.

The Lift Order Problem

With $k > 0$ and with M, V, and Q as above, determine whether ord(V, Q) $= k + 1$.

Note. In the present paper we also deal briefly with the related problem of determining whether every initial value problem (V, s, q) with q in Q has a solution lying entirely in Q, i.e., of determining whether V is *completely integrable over Q*. When ord(V, Q) $= k + 1$, we shall call *V k-suitable over Q*.

Rationale

In their classic paper on sprays, Ambrose *et al.* (1960) initially define sprays in terms of curves/flows in M. Then they give the formulation in terms of a stationary vector field V: $T(M) \rightarrow T^2(M)$ subject, in their notation, to $d\pi V(x) = x$, where x is in $T(M)$. The latter requirement is precisely what ensures that all solutions g to $y' = V(y)$ have the form $g = f'$, i.e., that the integral curves g for V all occur as lifts of curves in M. And it is this property, in turn, which establishes the relationship of sprays to connections, with the attendant meanings attaching to normal coordinates, geodesics, parallel translation, etc.

The same order issue is encountered (and resolved) in mechanics. Following the exposition in Abraham (1967), one begins, let us say, with a regular Lagrangian L: $T(M) \rightarrow$ R. Then, passing through a well-defined series of constructions, one eventually comes to a Hamiltonian vector field V_L: $T(M) \rightarrow T^2(M)$, and the equations of motion are given as $y' = V_L(y)$. The crucial step in the process comes in proving that, as with sprays, $d\pi V_L(x) = x$, where x is in $T(M)$. Again, the requirement is what ensures that solution curves g all arise from motion curves f in the configuration space M. In fact, we take it as a reasonable piece of general doctrine to insist that *any* dynamical system, defined through whatever combination of differential and integral mechanisms, must ultimately have its solutions represent curves in the base configuration manifold M.

As we will show, both in general theory and by example, when $k > 1$ it is *never* the case that a smooth (=completely integrable) vector field over $T^k(M)$ has all its solutions g of the form $g = f^{[k]}$. There are always unicorns among the horses, so to speak. Indeed, as we will see, there are always vectors x in $T^k(M)$ that are simply beyond the reach of any lifted curve $f^{[k]}$. In short, regardless of how one arrives at equations of motion $y' = V(t, y)$ over $T^k(M)$, with $k > 1$, if the equations are to be meaningful in the end, one has no alternative to the following program:

1. Specify subsets/submanifolds Q in $T^k(M)$—subuniverses of motion—relative to which V is completely integrable.
2. With Q as in 1, ensure that V is k-suitable relative to Q.

(In general, if V exhibits both properties 1 and 2 relative to Q, we shall call *V completely k-integrable relative to Q*.)

To my knowledge, not one paper in the existing journal literature deals with the order problem per se above the $k = 1$ level. It is my intention to present a few easily remembered principles governing the order problem in general, but to focus most attention on examples, since it is apparently a new field of inquiry. It is hoped that these examples will not be seen as pathological. While each has been constructed to illustrate something in particular, the examples represent normal behavior. Indeed, it is the exceptional higher differential level vector field all of whose differential lift solutions have the same (and appropriate) order. The approach to the subject matter in this paper is as follows:

1. We study the examples first by directly solving the relevant equations and then seeing how solutions behave in relation to the various submanifolds.
2. We develop a general theory of order.
3. We reexamine the examples in light of the general theory, to see how much of the behavior of solutions can be deduced *without* directly solving the equations.

Before proceeding to our first example, however, a great deal of needless calculation and possible conceptual misunderstanding can be avoided through an easy generalization of the curve lifting technique (a generalization we shall need in the general order theory anyway). Let f: $N \to M$ be any differentiable map of manifolds; let τ: $T(N) \to N$ be the tangent bundle projection; and let $(T^k(M), \pi_k)_{k\geq 0}$ be the full tangential resolution for M. Let

W: $N \to T(N)$ be any stationary vector field, i.e., $\tau \circ W = id_N$. We define the *W-lift of f* to be the map f'_W: $N \to T(M)$ given by $f'_W = T(f) \circ W$, where $T(f)$: $T(N) \to T(M)$ denotes the standard tangent map associated with f. Iterating this idea, with $f_W^{[0]} = f$, for $k \geq 0$, $f_W^{[k+1]}$: $N \to T^{k+1}(M)$ is given by $f_W^{[k+1]} = (f_W^{[k]})'_W$. We call $f_W^{[k]}$ the *kth-order differential lift of f relative to W.*

Note. When N is an open set in K, the global coordinate system allows us to write $T(N) = N \times K$, and τ is represented as a projection on the first factor. In this context, which is our primary concern in this paper, we may take W to be the *standard cross section*, i.e., $W(t) = (t, 1)$. Then our definition of $f_W^{[k]}$ coincides with our previous unsubscripted version.

Proposition 1. Let $1 \leq j \leq k$, and let g: $N \to T^k(M)$ and f: $N \to T^{k-j}(M)$ be differentiable maps related by $g = f_W^{[j]}$. Then f is unique with this property in relation to g: $f = \pi_{k-j} \circ \cdots \circ \pi_{k-1} \circ g$.

Proof. This is a straightforward induction on k, bearing in mind general commutativity for tangent maps, and will therefore not be written down.

In words, when g is a lift from a lower tangential level there is no doubt about what f it is the lift of: f is just the composite projection of g down to this level.

Example 1

Let V: $T(K) = K^2 \to T^2(K) = K^4$ be given by $V(x_0, x_1) = (x_0, x_1; x_1 - x_0, 0)$. Any solution $g = (g_0, g_1)$ to $y' = V(y)$ must satisfy

$$\begin{aligned} g'(t) &= (g_0(t), g_1(t); d/dt[g_0(t)], d/dt[g_1(t)]) \\ &= V(g_0(t), g_1(t)) \\ &= (g_0(t), g_1(t); g_1(t) - g_0(t), 0) \end{aligned}$$

Integration yields (1) $g_1(t) \equiv B$ and (2) $g_0(t) = B - Ae^{-t}$, where A and B are arbitrary constants.

Now, for a solution g as in (1) and (2) to be of the form $g = f'$, two things are required: $f(t) = B - Ae^{-t}$, by Proposition 1, and $d/dt[f(t)] = g_1(t) \equiv B$. These two conditions can hold simultaneously if and only if $A = B = 0$. Thus, in spite of the fact that V is completely integrable, the *only* lifted solution is the constant one: $g(t) \equiv (0, 0)$. In fact, if we consider V in relation to the punctured plane $Q = K^2\backslash\{(0.0)\}$, relative to which V is also completely integrable, V has no solutions in Q that are lifts.

We note in passing that there are many other constant solutions g to $y' = V(y)$ that are simply not lifts: $g(t) \equiv (B, B)$, where $B \neq 0$. Put differently, the connected zero-dimensional manifolds relative to which V is completely integrable are the diagonal singletons $\{(B, B)\}$, precisely one of which ($B = 0$) corresponds to a lifted solution. We also note that there are many submanifolds Q in $T(K)$ relative to which $\text{ord}(V, Q) = 2$ (i.e., V is 1-suitable over Q) *vacuously* (simply because there are no solutions g lying entirely in Q). For instance, let A in K be fixed and let $Q_A = \{(A, x_1): x_1 \neq A\}$. Indeed, all solutions g are *transversal* to Q_A.

To say that V as above lacks significance as a level-2 differential entity because of the behavior of the solutions to $y' = V(y)$ is not to say that V lacks either physical or geometric meaning as a first level entity in its own right. In fact, let $K = \mathrm{R}$, and let t in K be thought of as a time variable. Even though our solution set as in (1) and (2) constitutes a two-parameter family of distinct functions, two such functions $g = (B - Ae^{-t}, B)$ and $h = (B - Ce^{-t}, B)$ can nevertheless carve out the same orbit (= world line = submanifold).

Case 1. $A = C = 0$. In this case g and h are identical and "parametrize" the zero-dimensional manifold $\{(B, B)\} = B^{B,0}$, say.

Case 2. A and C are both >0. In this case g and h are related by $h(t + \mathrm{Ln}(C/A)) = g(t)$. Thus, g and h are but different parametrizations of the open horizontal ray lying to the left of the diagonal point (B, B), but approaching it with advancing time. Label this submanifold $g^{B,-}$.

Case 3. A and C are both <0. Again g and h are related by $h(t + \mathrm{Ln}(C/A)) = g(t)$. This time the functions parametrize the open horizontal ray to the right of (B, B) and approach it [i.e., the curves approach (B, B) from the right with advancing time]. Label the submanifold itself $g^{B,+}$.

Geometrically, therefore, V carves up the K^2 universe into mutually nonintersecting zero- and one-dimensional subuniverses $g^{B,-}$, $g^{B,0}$, or $g^{B,+}$. Since each of our integral curves g is maximal to begin with (because g is defined for all t), the totality of $g^{B,-}$, $g^{B,0}$, and $g^{B,+}$ for all B comprises the *phase portrait for V.*

Physically, V can be thought of as a generic capacity law (such as heating/cooling of an object placed in a constant-temperature = B bath).

Example 2

Let V: $T^2(K) = K^4 \rightarrow T^3(K) = K^8$ be given by

$$V(x_{00}, x_{01}, x_{10}, x_{11}) = (x_{00}, x_{01}, x_{10}, x_{11}; x_{01} + x_{11}, x_{11}, x_{11}, 0)$$

Then $g = (g_{00}, \ldots, g_{11})$ solves $y' = V(y)$ if and only if (1) $g_{11}(t) \equiv D$, (2) $g_{10}(t) = Dt + C$, (3) $g_{01}(t) = Dt + B$, and (4) $g_{00}(t) = Dt^2 + (B + D)t + A$, where A, B, C, and D are arbitrary parameters in K.

Case 1. $C \neq B + D$. Let $h = (g_{00}, g_{01})$ (the 1-level projection of g). Now $g = h'$ would require $d/dt[g_{00}(t)] = g_{10}(t)$ for all t, which is impossible since $C \neq B + D$. Thus, none of the solutions g arising in this case can be lifts from lower levels. Note the following curious fact, however: $h = (g_{00}, g_{01})$ as above has its own lift h' as a solution to $y' = V(y)$, even though $h' \neq g$. That is, $(h')' = V \circ h'$. (Example 1 shows dramatically that it is not generally the case that the lift of the projection of a solution is again a solution.)

Case 2. $C = B + D$, with $D \neq 0$. In this case, with $h = (g_{00}, g_{01})$, we find that $g = h'$ always holds. However, g is not of the form $g = (g_{00})^{[2]}$, since the latter would require $d^2/dt^2[g_{00}(t)] \equiv D \neq 0$, whereas $V(g(t))$ always has rightmost component $\equiv 0$.

Case 3. $C = B + D$, with $D = 0$. In this case, one easily verifies that $g = (g_{00})^{[2]}$ holds.

In summary, Example 2 presents a V having nontrivial solutions g of all possible orders.

Example 3

Let V: $K \times T(K) \to T^2(K)$ be a K-dependent field given by $V(t, (x_0, x_1)) = (x_0, x_1; 1 + t^2 - 2tx_0 + x_0^2, 0)$. We consider a solution $g = (g_0, g_1)$ to an initial value problem $(V, t_0, (x_0, x_1))$.

Case 1. $t_0 = x_0$. The solution is $g(t) = (t, x_1)$.

Case 2. $t_0 \neq x_0$. The solution is $g(t) = (t - 1/(t - t_0 + 1/(t_0 - x_0)), x_1)$.

Now, Case 2 yields no solutions g of the form $g = (g_0)'$, since $d/dt[g_0(t)]$ is nonconstant. On the other hand, Case 1 yields exactly one solution g with $g = (g_0)'$: this occurs when $x_1 = 1$. Consider the submanifold $Q = \{(x_0, 1)$: x_0 in $K\}$ in $T(K)$. V is completely integrable with respect to Q, since *all* solutions g meeting Q must lie entirely in Q, whether the solutions come by way of Case 1 or Case 2. But then V is *not* 1-suitable relative to Q, since not all these solutions are of the form $g = (g_0)'$. This is true, note, *even though* Q is fully parametrized by the one (maximal) integral curve g that does have the form $g = (g_0)'$.

Example 4

By way of contrast with Example 3, let V: $K \times T(K) \to T^2(K)$ be given by $V(t, (x_0, x_1)) = (x_0, x_1; t, 1)$. Easily, the general solution $g = (g_0, g_1)$ is given by $g(t) = (t^2/2 + A, t + B)$. Such a solution is of the form $g = (g_0)'$ if and only if $B = 0$. Or, letting $g^{A,B}(t) = (g_0^A(t), g_1^B(t))$, only the solutions $g^{A,0}$ are themselves lifts.

For any C in K, let $Q_C = \{(s^2/2 + C, s)$: s is in $K\}$. Easily, $g^{A,B}$ (or any restriction of $g^{A,B}$ to a nonempty open set in K) lies entirely in Q_C if and only if $A = C$ and $B = 0$. And, as in Example 3, the lifted solution $g^{C,0}$, which is maximal (it is defined over all of K), lies entirely in Q_C. Thus, V is indeed 1-suitable over Q_C, but V is dramatically *not* completely integrable over Q_C.

Example 5

Let V: $T^2(K) \to T^3(K)$ be given by

$$V(x_{00}, x_{01}, x_{10}, x_{11}) = (x_{00}, \ldots, x_{11}; x_{01}, 0, 0, 0)$$

Then, with $g(t) = (g_{00}(t), \ldots, g_{11}(t))$, $g'(t) = V(g(t))$ amounts to (1) $g_{11}(t) = D$, (2) $g_{10}(t) = C$, (3) $g_{01}(t) = B$, and (4) $g_{00}(t) = Bt + A$, where A, B, C, and D are arbitrary parameters in K. The solution to an initial value problem $(V, t_0, (x_{00}, \ldots, x_{11}))$ is given by (1′) $g_{11}(t) = x_{11}$, (2′) $g_{10}(t) = x_{10}$, (3′) $g_{01}(t) = x_{01}$, and (4′) $g_{00}(t) = x_{01}(t - t_0) + x_{00}$.

Observe that for g as in (1′)–(4′) to be of the form $g = (g_{00})^{[2]}$ it is necessary and sufficient that

$$x_{01} = x_{10} \qquad \text{and} \qquad x_{11} = 0 \tag{$*$}$$

We examine V in relation to five submanifolds $Q_1, \ldots, Q_5$ in $T^2(K)$ with regard both to the lift condition (∗) and to the general "solution-absorbing" properties of the submanifolds.

$Q_1 = \{(u, v, v, w)\colon w \neq 0\}$. From (1′)–(3′) it is apparent that $g(t_0)$ in Q_1 implies that $g(t)$ is in Q_1 for all t. Thus, V is completely integrable relative to Q_1. However, (∗) is *never* satisfied when $(x_{00}, \ldots, x_{11})$ is in Q_1. Thus, none of the solutions g meeting Q_1 is of the form $g = (g_{00})^{[2]}$.

$Q_2 = \{(0, v, v, w)\colon v \neq 0 \neq w\}$. This time any g solving the initial value problem with $(x_{00}, \ldots, x_{11})$ in Q_2 is *completely transient* relative to Q_2. That is, from (4′) and the fact that $x_{01} \neq 0$ we see that t_0 is the *only* argument value t for which $g(t)$ is in Q_2. Thus, vacuously, V is 2-suitable relative to Q_2: there simply are no solutions g lying entirely in Q_2 to consider regarding (∗).

$Q_3 = \{(u, v, v, 0)\colon u, v \text{ in } K\}$. As in the case of Q_1, V is completely integrable relative to Q_3. Moreover, (∗) holds whenever $(x_{00}, \ldots, x_{11})$ is in Q_3. Thus, V is completely 2-integrable relative to Q_3.

$Q_4 = \{(0, v, v, 0)\colon v \neq 0\}$. As with Q_2, any solution g to an initial value problem set over Q_4 is completely transient with regard to Q_4: t_0 is the only value for t such that $g(t)$ is in Q_4. Thus, again, V is vacuously 2-suitable relative to Q_4. This time, however, each g solving an initial value problem set over Q_4 *is* of the form $g = (g_{00})^{[2]}$, even though such a g is completely transient relative to Q_4.

$Q_5 = \{(u, v, 0, 0)\colon v \neq 0\}$. Clearly, an initial value solution g set over Q_5 has the property that $g(t)$ is in Q_5 for all t. Thus, V is completely integrable relative to Q_5. However, (*) is never satisfied when $(x_{00}, \ldots, x_{11})$ is in Q_5. Thus, V is extremely "un-2-suitable" relative to Q_5.

In summary, Example 5 reveals the astonishing variety of behaviors even a very simple "higher order" vector field V can exhibit, as regards both lifted solutions and solution-absorbing properties, for various submanifolds Q.

Example 6

Let V: $T^2(K) \to T^3(K)$ be given by

$$V(x_{00}, \ldots, x_{11}) = (x_{00}, \ldots, x_{11}; x_{01}, x_{10} - x_{01}, x_{11}, 0)$$

Any solution g to $y' = V(y)$ must take the form (1) $g_{11}(t) \equiv D$, (2) $g_{10}(t) = Dt + C$, (3) $g_{01}(t) = (C - D) + Dt + Be^{-t}$, and (4) $g_{00}(t) = A + (C - D)t + Dt^2/2 - Be^{-t}$, where A, B, C, and D are arbitrary parameters in K. In turn, the solution g to an initial value problem $(V, t_0, (x_{00}, \ldots, x_{11}))$ is given by

(1′) $g_{11}(t) \equiv x_{11}$

(2′) $g_{10}(t) = x_{11}(t - t_0) + x_{10}$

(3′) $g_{01}(t) = (x_{10} - x_{11}) + x_{11}(t - t_0) + (x_{01} - x_{10} + x_{11})\, e^{t_0 - t}$

(4′) $g_{00}(t) = x_{00} + (x_{11} - x_{10})(t_0 - t) + (t_0 - t)^2 x_{11}/2$
$+ (x_{10} - x_{01} + x_{11})(e^{t_0 - t} - 1).$

We consider V relative to two submanifolds Q_1, Q_2 in $T^2(K)$.

$Q_1 = \{(x_{00}, \ldots, x_{11})\colon x_{01} = x_{10} \text{ and } x_{11} \neq 0\}$. Observe that each solution curve g as in (1′)–(4′), where $(x_{00}, \ldots, x_{11})$ is in Q_1, meets Q_1 for only the one value of its argument: $t = t_0$. Thus, V is completely transient relative to Q_1 and hence it is 2-suitable relative to Q_1 vacuously.

$Q_2 = \{(x_{00}, \ldots, x_{11})\colon x_{01} = x_{10} \text{ and } x_{11} = 0\}$. This time, every solution g in the form (1′)–(4′), with $(x_{00}, \ldots, x_{11})$ in Q_2, lies entirely in Q_2 and also has the form $g = (g_{00})^{[2]}$. Thus, V is completely 2-integrable relative to Q_2.

In summary, Example 6 presents a situation in which V could not behave more antithetically in relation to two submanifolds Q_1 and Q_2. And yet, topologically, Q_2 = the boundary of Q_1, exclusive of Q_1 itself.

What the foregoing examples indicate about the order problem, at least by way of general disclaimer, is this: the order problem is not about smoothness (every function in sight is analytic); nor is it about local/global distinctions (each manifold has a global coordinate system, and each solution curve g is defined for all possible values of its argument variable t); nor is it about what one might otherwise anticipate as a harmonic interplay between manifold and boundary. We now turn to the development of a general order theory.

2. GENERAL ORDER THEORY

We return to the general differential lift context of Proposition 1, i.e., W: $N \to T(N)$ is a fixed stationary vector field, our "curves" are simply differentiable mappings defined on open sets in N, and our curve lifts are relative to W.

Let V: $N \times T^k(M) \to T^{k+1}(M)$ be an N-dependent $(k + 1)$-level vector field over $T^k(M)$, i.e., $(\pi_k \circ V)(t, y) = y$ always holds. By the *(ordinary) differential lift equation associated with W and V* we shall mean the formal expression $y'_W = V(t, y)$, a *solution* to which is a differentiable map g: $U \to T^k(M)$, with U open in N, such that $g'_W(t) = V(t, g(t))$ always holds. [We can formulate this in terms of the *graph of g*, gr(g), given by $\text{gr}(g)(t) = (t, g(t))$, by saying $g'_W = V \circ \text{gr}(g)$.] By an *initial value problem associated with W and V* we mean an expression (W, V, s, p), where s and p are chosen, respectively, from N and $T^k(M)$, a *solution* to which is a g solving $y'_W = V(t, y)$ such that $g(s) = p$. [All the terminology used previously—complete integrability, k-suitability, ord = ord(W, V, Q)—carries over into this more general context.]

Again, let $(T^k(M), \pi_k)_{k\geq 0}$ denote the full tangential resolution for the manifold M. As is well known, for $k \geq 1$, there is a multiplicity of "projections" $T^{k-i}(\pi_i)$ from $T^{k+1}(M)$ to $T^k(M)$, where $0 \leq i \leq k$, the tangent bundle projection being the one where $i = k$. We define two sequences of sets $({}_kM')_{k\geq 0}$ and $({}_kM)_{k\geq 0}$ as follows:

$${}_0M' = T(M)$$

and $(k \geq 1)$

$${}_kM' = \{x \text{ in } T^{k+1}(M)\text{: } 0 \leq i, j < k \text{ implies } T^{k-i}(\pi_i)(x) = T^{k-j}(\pi_j)(x)\}$$

Then let ${}_0M = M$; let ${}_1M = T(M)$; and let $(k \geq 1)$

$${}_{k+1}M = \{x \text{ in } {}_kM'\text{: } \pi_k(x) = T(\pi_{k-1})(x)\}$$

Proposition 2. 1. $(k \geq 0)$ π_k carries ${}_kM'$ (and its subset ${}_{k+1}M$) into ${}_kM$.

2. $(k \geq 1)$ $T(\pi_{k-1})$ carries ${}_kM'$ into ${}_{k-1}M'$.

3. Let f: $U \to M$ be a differentiable map and let $k \geq 0$. Then $T^{k+1}(f)$ carries ${}_kU'$ into ${}_kM'$, and it also carries ${}_{k+1}U$ into ${}_{k+1}M$.

4. Let W: $U \to T(U)$ be a differentiable cross section of τ_0, where $(T^k(U), \tau_k)_{k\geq 0}$ is the full tangential resolution for the manifold U. Let $k \geq 1$ and let $0 \leq i \leq k$. Then

$$T^{k-i}(\tau_i) \circ [T^k(W) \circ \cdots \circ W] = T^{k-1}(W) \circ \cdots \circ W$$

In particular, $[T^k(W) \circ \cdots \circ W]$ lies entirely in ${}_{k+1}U$.

5. With f as in statement 3 and with W as in statement 4, for all $k \geq 0$, $f^{[k]}_W$: $U \to T^k(M)$ lies entirely in ${}_kM$.

Proof. 1. We need only consider $k \geq 1$. Let i, j be arbitrary, with $0 \leq i, j < k$, and let x in $T^{k+1}(M)$ be such that $T^{k-i}(\pi_i)(x) = T^{k-j}(\pi_j)(x)$. Then

$$\begin{aligned} &T^{k-1-i}(\pi_i)((\pi_k)(x)) \\ &= (T^{k-1-i}(\pi_i) \circ \pi_k)(x) \\ &= (\pi_{k-1} \circ T^{k-i}(\pi_i))(x) \\ &= (\pi_{k-1} \circ T^{k-j}(\pi_j))(x) \\ &= T^{k-1-j}(\pi_j)(\pi_k(x)) \end{aligned}$$

Since i, j were arbitrary, the result follows immediately.

2. The argument is quite similar to that given in part 1.

3. This result follows from

$$\begin{aligned} T^{k-1}(f) \circ T^{k-1-i}(\tau_i) &= T^{k-1-i}(\pi_i) \circ T^k(f) \\ T^{k-1}(f) \circ T^{k-1-j}(\tau_j) &= T^{k-1-j}(\pi_j) \circ T^k(f) \end{aligned}$$

These each hold by the standard commutativity of tangent maps in relation to projections.

4. We first argue all cases where $i = k$. Then, with the result in hand for these cases, we argue the cases $0 \leq i < k$ by induction on k.

For the cases $i = k$, one has, by tangential commutativity,

$$\begin{aligned} &\tau_k \circ [T^k(W) \circ \cdots \circ W] \\ &= \tau_k \circ T[T^{k-1}(W) \circ \cdots \circ W] \\ &= [T^{k-1}(W) \circ \cdots \circ \tau_0] \circ W \\ &= T^{k-1}(W) \circ \cdots \circ W \circ (\tau_0 \circ W) \\ &= T^{k-1}(W) \circ \cdots \circ W \circ id_U \\ &= T^{k-1}(W) \circ \cdots \circ W \end{aligned}$$

as required.

For the cases $0 \leq i < k$, consider first $k = 1$:

$$T(\tau_0) \circ T(W) \circ W = T(\tau_0 \circ W) \circ W = T(id_U) \circ W = W$$

Now, inductively, assume the result for all $0 \leq i < k$ (bearing in mind that we separately know that the result holds for $i = k$ as well.) Consider $0 \leq i < k + 1$. We have

$$\begin{aligned} &T^{k+1-i}(\tau_i) \circ T^{k+1}(W) \circ \cdots \circ W \\ &= T[T^{k-i}(\tau_i) \circ T^k(W) \circ \cdots \circ W] \circ W \\ &= T[T^{k-1}(W) \circ \cdots \circ W] \circ W \\ &= T^k(W) \circ \cdots \circ W \end{aligned}$$

as required.

5. We need only consider $k \geq 1$. We can rewrite the inductive formulation of $f_W^{[k]}$ as $f_W^{[k]} = T^k(f) \circ T^{k-1}(W) \circ \cdots \circ W$. By part 4 of this proposition, $T^{k-1}(W) \circ \cdots \circ W$ lies entirely in ${}_kU$. Thus, by part 3 of this proposition, $T^k(f) \circ [T^{k-1}(W) \circ \cdots \circ W]$ lies entirely in ${}_kM$.

It is in the exact sense of part 5 of the proposition that we intended the remark in the Introduction (Rationale) to the effect that, for $k > 1$, most x in $T^k(M)$ are beyond the reach of curve lifts from M. For instance, in the finite-dimensional case, $\dim(T^k(M)) = 2^k \dim(M)$, whereas the subspace [in fact it is *always* a submanifold of $T^k(M)$] ${}_kM$ has $\dim({}_kM) = (k + 1)\dim(M)$.

Theorem 1. 1. ($k \geq 0$) Let W: $U \to T(U)$ be a tangent bundle cross section and let g: $U \to T^k(M)$ be differentiable, with g lying entirely in ${}_kM$. Then $T(g)$ lies entirely in ${}_kM'$. In particular, g'_W lies entirely in ${}_kM'$.

2. ($k \geq 1$) With W and g as in part 1, then g is of the form $g = f_W^{[k]}$ if and only if g'_W lies entirely in ${}_{k+1}M$. (In this case $f = \pi_0 \circ \cdots \circ \pi_{k-1} \circ g$, and g itself necessarily lies entirely in ${}_kM$.)

3. ($k \geq 1$) Let V: $N \times T^k(M) \to T^{k+1}(M)$ be an N-dependent differentiable vector field over $T^k(M)$, and let g: $U \to T^k(M)$, where U is open in N; solve $y'_W = V(t, y)$. Then g is of the form $g = f_W^{[k]}$ if and only if $V \circ \mathrm{gr}(g)$ lies entirely in ${}_{k+1}(M)$.

Proof. 1. We need only consider $k \geq 1$. Let $0 \leq i, j < k$. Then

$$\begin{aligned} & T^{k-i}(\pi_i) \circ T(g) \\ &= T(T^{k-1-i}(\pi_i) \circ g)) \\ &= T(T^{k-1-j}(\pi_j) \circ g)) \\ &= T^{k-j}(\pi_j) \circ g \end{aligned}$$

as required.

2. If $g = f_W^{[k]}$, then $g'_W = (f_W^{[k]})'_W = f_W^{[k+1]}$ lies entirely in ${}_{k+1}M$ by part 5 of Proposition 2. To see the converse (that g'_W lying entirely in ${}_{k+1}M$ implies g has the form $g = f_W^{[k]}$), we argue by induction on k.

$$k = 1: \qquad [\pi_0 \circ g]'_W = T(\pi_0 \circ g) \circ W = T(\pi_0) \circ (T(g) \circ W)$$

By the hypothesis of the theorem, the latter quantity

$$\begin{aligned} &= \pi_1 \circ (T(g) \circ W) = (\pi_1 \circ T(g)) \circ W = (g \circ \tau_0) \circ W \\ &= g \circ (\tau_0 \circ W) = g \circ id_U = g \end{aligned}$$

Inductively, assume the result for k and let g take its values in $T^{k+1}(M)$ with g'_W lying entirely in ${}_{k+2}M$.

We consider the one-step projection of g: $[\pi_k \circ g]$ taking its values in $T^k(M)$. By part 1 of Proposition 2, since g'_W lies entirely in ${}_{k+2}M \subseteq {}_{k+1}M'$, $\pi_{k+1} \circ g'_W = g$ lies entirely in ${}_{k+1}M$. But

$$[\pi_k \circ g]'_W = T(\pi_k) \circ T(g) \circ W = T(\pi_k) \circ g'_W$$

By the hypothesis on g_W, the latter quantity $= \pi_{k+1} \circ g'_W = g$, which, as we have just seen, lies entirely in ${}_{k+1}M$.

Thus, we may apply the induction hypothesis to $[\pi_k \circ g]$ and write

$$[\pi_k \circ g] = [\pi_0 \circ \cdots \circ \pi_{k-1} \circ [\pi_k \circ g]]_W^{[k]}$$

Now simply lift both sides of the latter equality to arrive at the desired form for g.

Note. The parenthetical remark, although we have in essence proved it, was already seen to be true in the context of the more general statement of Proposition 1.

3. Since $g'_W = V \circ \mathrm{gr}(g)$, this result is an immediate consequence of part 2 of the theorem.

Corollary. With general conditions as in the theorem, consider an initial value problem (W, V, t_0, x_0).

1. If x_0 is not in ${}_kM$, no solution g has the form $g = f_W^{[k]}$.
2. If $V(t_0, x_0)$ is not in ${}_kM'$, no solution g lies entirely in ${}_kM$ (whence, no solution g has the form $g = f_W^{[k]}$).
3. If $V(t_0, x_0)$ is not in ${}_{k+1}M$, then no solution g has the form $g = f_W^{[k]}$.
4. If V carries $N \times Q$ into ${}_{k+1}M$, then V is k-suitable relative to Q.
5. Assume V is completely integrable relative to Q. Then V is completely k-integrable relative to Q if and only if V carries $N \times Q$ into ${}_{k+1}M$.

Proof. 1. If g were of the form $g = f_W^{[k]}$, then, by the parenthetical remark in part 2 of the theorem, $g(t_0) = x_0$ would lie in ${}_kM$.

2. This follows from part 1 of the theorem, since $g'_W(t_0) = V(t_0, g(t_0)) = V(t_0, x_0)$

3. This follows from part 2 of the theorem, since, again, $g'_W(t_0) = V(t_0, g(t_0)) = V(t_0, x_0)$

4. Since any solution g satisfies $g'_W = V \circ \mathrm{gr}(g)$, the assumption, together with the assumption that g lies entirely in Q, would ensure that g'_W lies entirely in ${}_{k+1}M$. The result then follows from part 2 of the theorem.

5. By part 4 of the corollary, we need only show that V k-suitable relative to Q implies V carries $N \times Q$ into ${}_{k+1}M$. In this, the general assumption that V is completely integrable relative to Q is crucial. For, consider any (t_0, x_0) in $N \times Q$. We can solve the initial value problem (W, V, t_0, x_0) by means of a g lying entirely in Q, where (by k-suitability) g is of the form $g = f_W^{[k]}$. By

part 2 of the theorem, g'_W must lie entirely in ${}_{k+1}M$. In particular, $g'_W(t_0) = V(t_0, x_0)$ must lie in ${}_{k+1}M$.

Remarks. 1. Assertion 4 in the corollary cannot be made an if-and-only-if assertion without strong surrounding conditions as in assertion 5. Our Example 4 serves as a counterexample.

2. Observe that assertions 4 and 5 in the corollary (especially assertion 5) do *not* even require that Q be a submanifold of $T^k(M)$; Q is just a subset. However, as a practical matter, how can one ensure that a vector field V is completely integrable over a set with no additional differentiable structure?

3. ON GENERAL SOLVABILITY OVER SUBMANIFOLDS

To begin with, one has difficulties regarding submanifolds in the general Banach context that one does not have in finite dimensions. Even when Q_1 and Q_2 are Banach spaces, with Q_1 a (closed) subspace in Q_2, one cannot be sure that the sheaf of intrinsically differentiable functions on Q_1 coincides with the "pullback" sheaf from Q_2. That is, one cannot be sure that each differentiable function on an open set in Q_1 is locally expressible as restrictions of differentiable functions from the sheaf on Q_2. In practice, one needs a continuous linear projection L from Q_2 to Q_1 which serves as a left inverse to the inclusion map from Q_1 to Q_2. When Q_1 and Q_2 are Banach manifolds, the desired effect is achieved if the inclusion map from Q_1 to Q_2 has *local* differentiable left inverses. Let Q_1 and Q_2 be manifolds, therefore, with $\mathbf{i}$: $Q_1 \to Q_2$ denoting the (differentiable, we assume) inclusion map. We say Q_1 is *strongly embedded in* Q_2 if the following two conditions are met:

1. If U is a differentiable manifold and g: $U \to Q_2$ is a differentiable map, then h: $U \to Q_1$ is also differentiable, where h is just g considered as a map into Q_1, that is, $\mathbf{i} \circ h = g$.

2. $T(\mathbf{i})(T(Q_1))$ is itself a submanifold in $T(Q_2)$, with what corresponds to requirement 1 holding between $T(\mathbf{i})(T(Q_1))$ and $T(Q_2)$; and also $T(\mathbf{i})^{-1}$, considered as a map from $T(\mathbf{i})(T(Q_1))$ to $T(Q_1)$, is differentiable.

Note. In the finite-dimensional case, embedded implies strongly embedded, and in the general Banach case, the existence of local left inverses for $\mathbf{i}$ ensures a strong embedding.

In general, with Q_1 an embedded submanifold in Q_2 and with $\mathbf{i}$: $Q_1 \to Q_2$ denoting the inclusion map, let V: $N \times Q_2 \to T(Q_2)$ be a differentiable N-dependent vector field over Q_2. We say that V is *closed relative to* Q_1 if $V(N \times Q_1)$ is contained in $T(\mathbf{i})(T(Q_1))$.

Theorem 2. Let Q_1 be strongly embedded in Q_2, with $\mathbf{i}$ denoting the inclusion map. Let g: $U \to Q_2$ be differentiable. Let V: $N \times Q_2 \to T(Q_2)$ be an N-dependent differentiable vector field over Q_2.

1. g lies entirely in Q_1 if and only if g'_W lies entirely in $T(\mathbf{i})(T(Q_1))$, where $W: U \rightarrow T(U)$ is any differentiable cross section of the tangent bundle map $T(U) \rightarrow U$.

2. With W as in part 1, let U be open in N and let $g: U \rightarrow Q_2$ solve $y'_W = V(t, y)$. Then g lies entirely in Q_1 if and only if $V \circ \text{gr}(g)$ lies entirely in $T(\mathbf{i})(T(Q_1))$.

Note. In parts 3 and 4 of the theorem, we assume N is an open set in K and that W is the standard cross section.

3. V is completely integrable relative to Q_1 if and only if V is closed relative to Q_1.

4. Assume Q_2 is itself an embedded submanifold in $T^k(M)$. Then V is completely k-integrable relative to Q_1 if and only if $V(N \times Q_1)$ is contained in the intersection of ${}_{k+1}M$ and $T(\mathbf{i})(T(Q_1))$.

Proof. 1. If g lies entirely in Q_1, then g can be realized as $g = \mathbf{i} \circ h$, where $h: U \rightarrow Q_1$ is differentiable. But certainly $h'_W = T(h) \circ W$ lies in $T(Q_1)$. Thus,

$$g'_W = T(\mathbf{i} \circ h) \circ W = T(\mathbf{i}) \circ (T(h) \circ W) = T(\mathbf{i}) \circ h'_W$$

lies in $T(\mathbf{i})(T(Q_1))$.

Conversely, let g'_W lie entirely in $T(\mathbf{i})(T(Q_1))$. Let $\pi: T(Q_2) \rightarrow Q_2$ and $\rho: T(Q_1) \rightarrow Q_1$ denote, respectively, the tangent bundle projections. For each t in U, let $w(t)$ denote the (necessarily unique) element in $T(Q_1)$ such that $T(\mathbf{i})(w(t)) = g'_W(t) = T(g)(W(t))$. Let $h(t) = \rho(w(t))$. Then

$$(\mathbf{i} \circ h)(t) = \mathbf{i}(\rho(w(t))) = \pi(T(\mathbf{i})(w(t))) = \pi(g'_W(t)) = g(t)$$

Thus, $g(t)$ must lie in Q_1.

2. This follows immediately from part 1 of the theorem, since $g_W' = V \circ \text{gr}(g)$.

3. Assume that V is completely integrable relative to Q_1, and consider (relative to Q_2) an initial value problem (W, V, t_0, q_1), where q_1 is in Q_1. By assumption, there is a (necessarily unique, by smoothness) $g: U \rightarrow Q_2$ solving the initial value problem with g lying entirely in Q_1. Let $h: U \rightarrow Q_1$ be the differentiable map such that $\mathbf{i} \circ h = g$. Then

$$T(\mathbf{i})(h'(t_0)) = (\mathbf{i} \circ h)'(t_0) = V(t_0, g(t_0)) = V(t_0, q_1)$$

So the necessity of closure is clear.

Conversely, assume that the closure criterion is met. Define a smooth vector field $X: N \times Q_1 \rightarrow T(Q_1)$ by $X = T(\mathbf{i})^{-1} \circ V$, where V is restricted to $N \times Q_1$. By the fundamental existence/uniqueness theorem, since X is smooth, X is completely integrable over Q_1. Let h solve any initial value problem (X,

t_0, q_1). Then, easily, $g = \mathbf{i} \circ h$ solves (V, t_0, q_1), and g lies entirely in Q_1. In short, V is indeed completely integrable relative to Q_1.

4. This result follows immediately from part 3 of this theorem and from part 5 of the corollary to Theorem 1.

Remark. In parts 3 and 4 of Theorem 2, we have not made the most general assertions possible, because we have specialized the context. However, it is the context of primary interest where curves are concerned. And a broader spatial assumption for the independent variable t, together with alternative cross section W, would oblige us to resort to the Frobenius symmetry theorem in place of the smoothness-based ordinary existence theorem.

4. INVOLUTIONS AND COMPLETE VECTOR FIELD LIFTS

Since its introduction by Kobayashi (1957) the general differential field-lifting technique, based upon second-order involution, has been the object of study or use by numerous investigators. The specific problem is the lifting of fields of both tensor and nontensor type by means of tangent maps in a way that preserves type and standard operations (such as the Lie bracket.) The general problem was what motivated Bowman (1970a,b) to introduce the notion of a *restricted* (= canonical, in the language of Bowman) tangential resolution of a manifold M. Indeed, it was the desire to place such constructions on a firm Kleinian group representational context that motivated Bowman and Pond (1975). The present purpose, however, is to examine the technique as it applies to stationary vector fields from the standpoint of the differential lift equations generated, focusing especially on their order aspects.

We review the definition and most basic properties of a (second-order) involution. Let P be any differentiable manifold and let ρ: $T(P) \to P$ and μ: $T^2(P) \to T(P)$ denote the respective tangent bundle projections. One has a diffeomorphism I_P: $T^2(P) \to T^2(P)$ given, in local coordinates, by

$$I_P([T(\theta), x_{00}, x_{01}, x_{10}, x_{11}]) = [T(\theta), x_{00}, x_{10}, x_{01}, x_{11}]$$

Then I_P satisfies $I_P \circ I_P = id_{T^2(P)}$ (the reason for calling I_P an *involution*); $I_P \circ \mu = T(\rho)$; and, in our language, x is in ${}_2P$ if and only if $I_P(x) = x$.

Now let $1 \leq j \leq k$. Let M be any differentiable manifold with full tangential resolution $(T^n(M), \pi_n)_{n \geq 0}$. Then one has an involution

$$T^{k-j}(I_{T^{j-1}(M)}): \quad T^{k+1}(M) \to T^{k+1}(M)$$

satisfying the structural equation

$$T^{k-j+1}(\pi_{j-1}) \circ T^{k-j}(I_{T^{j-1}(M)}) = T^{k-j}(\pi_j) \tag{*}$$

Thus, we have a multiplicity of maps $T^{k+1}(M) \to T^{k+1}(M)$, just as we have a multiplicity of "projections" $T^{k+1}(M) \to T^{k}(M)$. They are strongly interrelated, as shown in the following result.

Theorem 3. Let $k \geq 1$. Then:

1. For $1 \leq j \leq k$,

$$T^{k-j}(I_{T^{j-1}(M)})(x) = x$$

if and only if

$$T^{k-j+1}(\pi_{j-1})(x) = T^{k-j}(\pi_j)(x)$$

2. x is in ${}_{k+1}M$ if and only if

$$x = I_{T^{k-1}(M)}(x) = \cdots = T^{k-1}(I_M)(x)$$

3. Let V: $M \to T(M)$ be a stationary differentiable vector field. Let

$$V_k = I_{T^{k-1}(M)} \circ \cdots \circ T^{k-1}(I_M) \circ T^k(V): \quad T^k(M) \to T^{k+1}(M)$$

Then V_k is itself a vector field [called *the complete lift of V to $T^k(M)$*].

4. Let f: $U \to M$ be a differentiable map, where U is open in the differentiable manifold N. Let W: $T(N) \to N$ be a differentiable cross section of the tangent bundle map. Then:

(a) f solves $y'_W = V(y)$ if and only if
(b) $f_W^{[k]}$ solves $u'_W = V_k(u)$ if and only if
(c) f solves $v_W^{[k+1]} = [T^k(V) \circ \cdots \circ T(V) \circ V](v)$.

Proof. 1. Assuming $T^{k-j}(I_{T^{j-1}(M)})(x) = x$, simply apply the structural equation (*) to yield $T^{k-j+1}(\pi_{j-1})(x) = T^{k-j}(\pi_j)(x)$. [*Note:* The equation (*) itself is merely the result of successive applications of T to the basic commutativity relationship among $I_{T^{j-1}(M)}$, $T(\pi_{j-1})$, and π_j.]

The converse argument is more complicated. First observe that all maps involved leave the base point in M unmoved. Then, since involution (and, hence, all higher tangential versions of it) is preserved under differentiated coordinate maps (indeed, it was this phenomenon that allowed involution to be defined as a global manifold map in the first place), it suffices to replace M by the Banach space F upon which M is modeled and prove the result there.

Let $(T^n(F), \alpha_n)_{n \geq 0}$ denote the full tangential resolution for the Banach space F, where, owing to the global coordinate system for F, we can treat $T^{n+1}(F)$ as $T^n(F) \times T^n(F)$ always.

In addition to being bundle spaces, the various $T^n(F)$ can also be viewed as Banach spaces. In this regard, *all* the various projections, involutions, and their higher tangential extensions are continuous linear maps. And for any continuous linear map L, one has the fact that $T(L)(x; y) = (L(x); L(y))$. We simply use this fact on the relevant maps in an inductive argument on $k \geq 1$.

$k = 1$: In this case $j = 1$ as well, and the assertion to be proved is: $I_F(x) = x$ if and only if $T(\alpha_0)(x) = \alpha_1(x)$, which obviously holds.

Assume the result holds for k, inductively, and consider $1 \leq j \leq k + 1$. Now for $j = k + 1$, the assertion is just that $I_{T^k(F)}(x) = x$ if and only if $T(\alpha_k)(x) = \alpha_{k+1}(x)$, which, again, is just one of the basic facts about second-order involutions. Thus, we are left with only the cases $1 \leq j \leq k$ to consider.

It is convenient to write x in split form: $x = (x_0; x_1)$. Bearing the general observation in mind about L above, we are left with three quantities to compare:

(A) $T^{k+1-j+1}(\alpha_{j-1})(x) = (T^{k-j+1}(\alpha_{j-1})(x_0); T^{k-j+1}(\alpha_{j-1})(x_1))$
(B) $T^{k+1-j}(\alpha_j)(x) = (T^{k-j}(\alpha_j)(x_0); T^{k-j}(\alpha_j)(x_1))$
(C) $T^{k+1-j}(I_{T^{j-1}(F)})(x) = (T^{k-j}(I_{T^{j-1}(F)})(x_0); T^{k-j}(I_{T^{j-1}(F)})(x_1))$

But, bearing in mind the induction hypothesis, in relation to *each* coordinate position on the right in (A), (B), and (C), it is evident that $A = B$ implies that

$$T^{k+1-j}(I_{T^{j-1}(F)})(x) = (x_0, x_1) = x$$

which completes the induction.

2. This result follows immediately from part 1.

3. We need to show that $\pi_k \circ V_k = id_{T^k(M)}$. This is a straightforward chain of reductions using the structural equations (*):

$$\begin{aligned}
&\pi_k \circ I_{T^{k-1}(M)} \circ \cdots \circ T^{k-1}(I_M) \circ T^k(V) \\
&= T(\pi_{k-1}) \circ T(I_{T^{k-2}(M)}) \circ \cdots \circ T^{k-1}(I_M) \circ T^k(V) \\
&= \cdots \\
&= T^{k-1}(\pi_1) \circ T^{k-1}(I_M) \circ T^k(V) = T^k(\pi_0) \circ T^k(V) \\
&= T^k(\pi_0 \circ V) = T^k(id_M) = id_{T^k(M)}
\end{aligned}$$

as required.

4. As in part 3 above, we prefer to argue the matter informally as a series of successive reductions/extensions rather than as a formal inductive argument.

(c) implies (a): Assume $f_W^{[k+1]} = T^k(V) \circ \cdots \circ T(V) \circ V \circ f$. By Theorem 1, we know that $f_W^{[k+1]}$ lies entirely in ${}_{k+1}M$. Thus, we can apply π_k or $T^k(\pi_0)$ to $f_W^{[k+1]}$ with the same effect. Apply π_k to the left side of the equation and $T^k(\pi_0)$ to the right side. The result is the reduction $f_W^{[k]} = T^{k-1}(V) \circ \cdots \circ V \circ f$. In a similar fashion, apply π_{k-1} and $T^{k-1}(\pi_0)$ to reduce the level of the equation a step further. Continuing in this manner, one eventually arrives at the desired result: $f'_W = V \circ f$.

(a) implies (c): We begin with $f_W^{[1]} = V \circ f$. First apply T, and then apply W on the extreme right of each member of the resulting equation. One obtains $f_W^{[2]} = T(V) \circ T(f) \circ W$, which in turn $= T(V) \circ f_W^{[1]} = T(V) \circ V \circ f$. Continue to apply T and W in this manner. We eventually arrive at the result $f_W^{[k+1]} = T^k(V) \circ \cdots \circ T(V) \circ F \circ f$.

(b) implies (a): We start with $f_W^{[k+1]} = V_k \circ f_w^{[k]}$. Bearing in mind that, from part 2 of the theorem, applying $I_{T^{k-1}(M)}$ to $f_W^{[k+1]}$ has no effect since $f_W^{[k+1]}$ lies in ${}_{k+1}M$, and also that $I_{T^{k-1}(M)}$ is involutive, apply $I_{T^{k-1}(M)}$ to our starting equation. Collecting terms appropriately, we obtain the result $f_W^{[k+1]} = T(V_{k-1} \circ f_W^{[k-1]}) \circ W$. Let τ: $T(U) \to U$ denote the tangent bundle projection. Next, bearing in mind general tangential commutativity, apply π_k to both sides of the equation immediately above. Since $\tau \circ W = id_U$, the result is the reduction $f_W^{[k]} = V_{k-1} \circ f_W^{[k-1]}$. Next apply $I_{T^{k-2}(M)}$ and π_{k-1} in succession (if necessary) to reduce the level of the equation one step further. Continuing in this manner, one eventually arrives at the result $f'_W = V \circ f$.

(a) implies (b): Start with $f'_W = V \circ f$. Execute the following three-step procedure on the equation: (i) apply T; (ii) apply W on the extreme right of each member of the resulting equation; (iii) apply the appropriate I-involution to the equation resulting from (ii). The overall result is a simultaneous lift of both equation and solution: $f_W^{[2]} = V_1 \circ f_W^{[1]}$. Simply continue to apply steps (i)–(iii) repeatedly to arrive at the end result: $f_W^{[k+1]} = V_k \circ f_W^{[k]}$.

Major Remark. Observe that part 4 of Theorem 3 merely asserts the *equivalence* of three sets of equations for a certain class of functions. The result is virtually meaningless, *in the absence* of deeper information on the existence of solutions to initial value problems in any of the three contexts. It is just such information that the general order theory (in particular, parts 3 and 4 of Theorem 2) was set up to provide. We shall pursue the analysis from the point of view of the order problem, partly for its own sake, but *mainly* as an illustration of how to proceed with an order analysis in other applied settings, such as higher order dynamical systems (especially in mechanics: see the conclusion in Section 6 for an overall description of the process). First we shall need a result of some interest outside the immediate context.

Definition. With $k \geq 1$, let Q be the set of all x in $T^k(M)$ such that, for some (necessarily unique) x_0 in M, $x = [T^{k-1}(W) \circ \cdots \circ W](x_0)$.

Proposition 3. For $k \geq 1$:

1. $T^k(V)(x)$ is in ${}_{k+1}M$ if and only if x is in Q.
2. $Q = \{x \text{ in } T^k(M)\text{: } V_k(x) \text{ is in } {}_{k+1}(M)\}$.

Proof. 1. If $x = (T^{k-1}(V) \circ \cdots \circ V)(x_0)$ in Q, then $T^k(x)$ is in ${}_{k+1}M$ by part 4 of Proposition 2.

We argue the converse by induction on k.

$k = 1$: Suppose $T(V)(x)$ is in ${}_2M$. Then

$$x = T(\pi_0 \circ V)(x) = T(\pi_0)(T(V)(x)) = \pi_1(T(V)(x)) = (V \circ \pi_0)(x)$$

which is clearly in $V(M)$, as required.

Inductively, assume the result for k, and suppose $T^{k+1}(V)(x)$ is in ${}_{k+2}M$. Then

$$\begin{aligned} x &= T^{k+1}(\pi_0 \circ V)(x) \\ &= T^{k+1}(\pi_0)(T^{k+1}(V)(x)) \\ &= \pi_{k+1}(T^{k+1}(V)(x)) \end{aligned}$$

which is in ${}_{k+1}M$ by part 1 of Proposition 2. But

$$\pi_{k+1}(T^{k+1}(V)(x)) = T^k(V) \circ \pi_k(x) = T^k(V)(\pi_k(x))$$

So, by the inductive hypothesis, $\pi_k(x) = (T^{k-1}(V) \circ \cdots \circ V)(x_0)$, say, whence

$$x = T^k(V)(\pi_k(x)) = (T^k(V) \circ \cdots \circ V)(x_0)$$

and the induction is complete.

2. Let $y = V_k(x)$ be in ${}_{k+1}M$. Simply apply the composite involution to y in reverse order to conclude that $y = T^k(V)(x)$, which is assumed to be in ${}_{k+1}M$. Thus x itself must be in Q by part 1 of this proposition.

Conversely, let x be in Q. Then, by part 4 of Proposition 2, $T^k(V)(x)$ is in ${}_{k+1}M$. But then, by part 2 of Theorem 3, $V_k(x) = T^k(V)(x)$. Thus, $V_k(x)$ is indeed in ${}_{k+1}M$, as required.

We now proceed with an order analysis of V_k in relation to Q, drawing on parts 2 and 3 of Theorem 3 for the structure of V_k, upon part 2 of Proposition 3 for the essential structure of Q, and upon part 4 of Theorem 2 as our main tool from general order theory. *Note:* We assume, from this point on in the analysis, that N is an open set in K and that W is the standard tangential cross section there: $W(t) = (t, 1)$. We label the analysis:

Example 7

1. Solutions g to $y' = V_k(y)$ lying entirely in Q are the only ones *possible* of the form $g = f^{[k]}$.

Proof. This follows from part 2 of Proposition 3 and from the corollary to Theorem 1.

2. Q can be regarded as a strongly embedded submanifold in $T^k(M)$.

Proof. Let $\mathbf{i}$: $Q \to T^k(M)$ denote the inclusion map. Then

$$T^{k-1}(V) \circ \cdots \circ V \circ \pi_0 \circ \cdots \circ \pi_{k-1}: \quad T^k(M) \to T^k(M)$$

is a globally defined, differentiable map that serves as a left inverse for $\mathbf{i}$.

3. V_k and $T^k(V)$ coincide on Q.

Proof. By Proposition 2, part 4, $T^k(V) \circ \cdots \circ V$ lies entirely in ${}_{k+1}M$. Thus, by part 2 of Theorem 3, successive applications of the various involutions at this level can have no further effect. The result follows immediately.

4. V_k is k-suitable relative to Q.

Proof. By result 3 above, $V_k(Q) = T^k(V)(Q)$. But, again by part 4 of Proposition 2 and the definition of Q, we know that $T^k(V)(Q)$ is a subset of ${}_{k+1}M$. Thus, $V_k(Q)$ is contained in ${}_{k+1}M$, and the result follows by (the stationary version of) part 4 of the corollary to Theorem 1.

5. V_k is closed relative to Q.

Proof. Consider the embedding map $T^{k-1}(V) \circ \cdots \circ V$: $M \to T^k(M)$. This carries M (diffeomorphically) onto the submanifold Q in $T^k(M)$. Let h: $M \to Q$ be the diffeomorphism such that $\mathbf{i} \circ h = T^{k-1}(V) \circ \cdots \circ V$. Then we have

$$\begin{aligned} &T(\mathbf{i})(T(Q)) \\ &\quad = T(\mathbf{i})(T(h)(T(M))) \\ &\quad = T(\mathbf{i} \circ h)(T(M)) \\ &\quad = (T^k(V) \circ \cdots \circ T(V))(T(M)) \end{aligned}$$

Now, $V(M)$ is contained in $T(M)$. Thus, from the preceding computation of $T(\mathbf{i})(T(Q))$, we see that $T(\mathbf{i})(T(Q))$ contains $[T^k(V) \circ \cdots \circ T(V) \circ V](M)$. That is, $T(\mathbf{i})(T(Q))$ contains $T^k(V)(Q)$. But, by result 3 above, $T^k(V)$ and V_k coincide on Q. In summary, therefore, $T(\mathbf{i})(T(Q))$ contains $V_k(Q)$; that is, V_k meets the closure criterion in relation to Q.

6. V_k is completely k-integrable relative to Q.

Proof. This follows immediately from items 2, 4, and 5 above, and part 4 of Theorem 2.

This concludes our order analysis for Example 7. Observe that we have two significant pieces of information now that we did not have simply as a result of the manipulations in part 4 of Theorem 3:

(a) V_k is completely integrable relative to the strongly embedded submanifold Q in $T^k(M)$.
(b) A solution g to $u' = V_k(u)$ has the form $g = f^{[k]}$ if and only if g lies entirely in Q.

If we were in the context of a higher order dynamical system (especially in mechanics), the order study phase would at this point be regarded as complete, and one could (with a justifiable sense of confidence in the overall meaning of the model) proceed with qualitative studies (periodicity, various questions of stability) and asymptotic studies on V_k in relation to Q. We return to the dynamical application in Section 6.

There is one further tool/technique useful in the analysis of a *stationary* vector field V: $T^k(M) \to T^{k+1}(M)$. Namely, given a nonempty subset P in $T^k(M)$, by the *k-crucial set for V in P* we shall mean the subset of P, to be denoted P^V, consisting of all x in P for which $V(x)$ is in ${}_{k+1}M$. [P^V could turn out to be empty, of course. But, looking back at the involution context and part 2 of Proposition 3, note the vital role played in the order analysis by the fact that Q turned out to be $(T^k(M))^{V_k}$. Indeed, in any higher order dynamical study of a vector field V defined on $T^k(M)$, lacking inspiration or insider information, the most natural candidate the investigator might wish to examine in relation to V is the subset $Q = (T^k(M))^V = ({}_kM)^V$.]

It is by no means automatic that P^V will turn out to be a strongly embedded submanifold in $T^k(M)$, as happened in Example 7. But observe that, by Theorem 1, any solution g to $y'_W = V(y)$ that lies entirely in P and is of the form $g = f^{[k]}_W$ must also lie entirely in the subset P^V. Now suppose P^V *happens* to be a (strongly) embedded submanifold in $T^k(M)$. Then the solution g as above can be expressed $g = \mathbf{i} \circ h$, where h is differentiable into P^V. But then $g'_W = T(\mathbf{i}) \circ h'_W$. In summary, g must lie entirely in P^V and g'_W must lie entirely in $T(\mathrm{i})(T(P^V))$. By the same token, $g'_W = V \circ g$ must lie in $V(P^V)$. Thus, one would hope to make progress in any order analysis by studying the structure of the intersection of $V(P^V)$ with $T(\mathrm{i})(T(P^V))$. In particular, this is useful when $P = {}_kM$.

5. EXAMPLES 1–6 REVISITED

Note that, by an easy calculation,

$${}_2K = \{(x_{00}, \ldots, x_{11}): x_{01} = x_{10}\}$$

$${}_3K = \{(x_{000}, \ldots, x_{111}): x_{001} = x_{010} = x_{100} \text{ and } x_{011} = x_{101} = x_{110}\}$$

[Indeed, using a binary multiindexing scheme for coordinate values as above, a relatively easy inductive argument leads to the following result: ${}_kK$ consists of all $(x_{0\cdots 0}, \ldots, x_{1\cdots 1})$ such that $x_{i_1\cdots i_k} = x_{j_1\cdots j_k}$ whenever $i_1 + \cdots + i_k = j_1 + \cdots + j_k$.]

Example 1

$T(K)^V = \{(0, x): x \text{ is in } K\}$, which is certainly a (strongly) embedded submanifold in $T(K)$. By definition, $V(T(K)^V)$ is contained in ${}_2K$. However, $V(T(K)^V) = \{(0, x, x, 0): x \text{ in } K\}$, which meets $T(\mathbf{i})(T(T(K)^V)) = \{(0, x, 0,$

y): x, y are in $K\}$ only in the singleton $\{(0, 0, 0, 0)\}$. Thus, from the previous reasoning on k-crucial sets, the only possible candidate for a set relative to which V is 1-suitable is $Q_{0,0} = \{(0, 0)\}$. And, indeed, $V(Q_{0,0})$ is contained in the intersection of $T(\mathbf{i})(T(Q_{0,0}))$ with ${}_2K$—whence, by Theorem 2, V is completely 1-integrable relative to $Q_{0,0}$. In summary, the only solutions g to $y' = V(y)$ of the form f' are those lying entirely in $Q_{0,0}$.

On the other hand, we can use general order analysis to find all *constant* solutions to $y' = V(y)$. Namely, let (x, y) be arbitrary in $T(K)$ and consider the singleton $\{(x, y)\} = Q_{x,y}$. As with any singleton manifold, $T(\mathbf{i})(T(Q_{x,y})) = \{(x, y, 0, 0)\}$. Thus, to fulfill the requirements of closure (Theorem 2, part 3), we are looking for the singular points (x, y) of V. Easily, $V(x, y) = (x, y, 0, 0)$ if and only if $x = y$. Thus, the constant solutions $g = (g_0, g_1)$ to $y' = V(y)$ are all g of the form $g \equiv (B, B)$, say, where B in K is arbitrary. (As we have seen, one of these, when $B = 0$, represents the only solution g of the form $g = g_0'$.)

Perhaps somewhat unfairly (but not by much), one realizes from the algebra of V that *any* solution $g = (g_0, g_1)$ to $y' = V(y)$ requires $d/dt[g_1(t)] \equiv 0$, whence g_1 is necessarily constant. Thus, V cannot be completely integrable relative, say, to the trace of any K-curve that is not a subset of the diagonal, but permits some degree of continuous variation in the second variable. Thus, for complete integrability, one is led to examine, for instance, submanifolds of the form $Q_{I,B} = \{(x, B)$: x is in $I\}$, where I is open in K, and where B is an arbitrary constant in K. Easily, $T(\mathbf{i})(T(Q_{I,B})) = \{(x, B, y, 0)$: x in I, y in $K\}$, which clearly contains $V(Q_{I,B})$. Thus, by the closure criterion, V is completely integrable relative to $Q_{I,B}$. Of course, no such $Q_{I,B}$ is 1-suitable, since it contains points (and, therefore, initial value solutions) other than (0, 0).

Example 2

Here, $(T^2(K))^V = \{(u, v, v, 0)$: u and v are in $K\}$. Then

$$T(\mathbf{i})(T(T^2(K)^V)) = \{(u, v, v, 0; x, y, y, 0): u, v, x, \text{ and } y \text{ are in } K\}$$

and $V((T^2(K))^V)$ is surely contained in the latter set. Thus, V is closed (= completely integrable) relative to $(T^2(K))^V$. Then the only *possible* solutions g of the form $g = f^{[2]}$ can be written

$$\begin{aligned} g(t) &= (g_0(t), g_1(t), g_1(t), 0) \\ &= (g_0(t), d/dt[g_0(t)], d/dt[g_0(t)], d^2/dt^2[g_0(t)]) \end{aligned}$$

One notes, not unfairly, that $g = g_0^{[2]}$ requires $d^2/dt^2[g_0(t)] \equiv 0$, whence $g_0(t) = Rt^2 + St + U$, say. Pressing further, then, $d/dt[g_0(t)] = 2Rt + S = g_1(t)$ forces $2R = d/dt[g_1(t)] \equiv 0$, whence $R = 0$. Thus, there being no other constraints, it must be the case that $g = g_0^{[2]}$ happens precisely when $g_0(t) =$

$St + U$. In summary, we have been led (correctly) to case 3 in the example primarily just from order considerations.

Example 3

Let $g = (g_0, g_1)$ represent any solution to $y' = V(t, y)$. We note (again, not unfairly) that $d/dt[g_1(t)] \equiv 0$ forces $g_1(t) \equiv B$, say, where B is a constant in K. Thus g must lie entirely in the submanifold $Q_B = \{(x, B): x \text{ in } K\}$. Now $T(\mathrm{i})(T(Q_B)) = \{(x, B, y, 0): x \text{ and } y \text{ in } K\}$, which clearly contains $V(K \times Q_B)$. Thus, V is closed (= completely integrable) relative to Q_B.

On the other hand, $V(K \times Q_B)$ is far from being contained in ${}_2K$, whence (since V *is* completely integrable over Q_B) V is not 1-suitable over Q_B. Nevertheless, we can use the order machinery to analyze the putative solution g more deeply. By Theorem 1 (or its corollary) we must have $V(t, g(t))$ always in ${}_2K$ for g to be of the form $g = g_0'$. That is, we must have:

(i) $1 + (t - g_0(t))^2 \equiv B$.

At the same time $d/dt[g_0(t)] = g_1(t) \equiv B$ demands that:

(ii) $g_0(t) = Bt + A$, say.

Substituting the form (ii) into equation (i) yields the requirement:

(iii) $1 + (t - [Bt + A])^2 \equiv B$, which can happen if and only if $B = 1$ and $A = 0$.

Thus, led by order considerations, we have narrowed the possibilities for a solution to have the form $g = g_0'$ down to a single function: $g(t) = (t, 1)$. One verifies directly that this g does indeed satisfy the original equation.

In summary:

(a) For $B \neq 1$, V is not 1-suitable over Q_B, because it *is* completely integrable there, but does not contain our solution $g(t) = (t, 1)$.

(b) For $B = 1$, even though Q_B does contain $g(t) = (t, 1)$, V is still not 1-suitable over Q_B, since Q_B contains other solutions to initial value problems as well, namely, $(V, t_0, (x_0, 1))$, where $t_0 \neq x_0$.

Example 4

We note that, if $g = (g_0, g_1)$ represents any solution to $y = V(t, y)$, then $d/dt[g_1(t)] \equiv 1$ forces $g_1(t) = t + B$, say. But $V(t, (g_0(t), t + B)) = (g_0(t), t + B, t, 1)$ lies entirely in ${}_2K$ if and only if $B = 0$. Thus, if g is to have the form $g = g_0'$, then $g_1(t) = t$ is forced. Continuing, then, $d/dt[g_0(t)] = g_1(t)$ forces $g_0(t) = t^2/2 + A$, say. Finally, one verifies directly that, for such a g, $g'(t) = V(t, g(t))$. Thus, we have easily found all solutions to $y' = V(t, y)$ having the form $g = g_0'$: they comprise a one-parameter (A) family $g(t) = (t^2/2 + A, t)$.

Now consider any of the solution manifolds $Q_A = \{(r^2/2 + A, r): r \text{ is in } K\}$. One calculates that

$$T(\mathbf{i})(T(Q_A)) = \{(r^2/2 + A, r, 2ry, y): r \text{ and } y \text{ in } K\}$$

Thus, $V(t, (s^2/2 + A, s)) = (s^2/2 + A, s, t, 1)$ is in $T(\mathbf{i})(T(Q_A))$ if and only if $t = 2s$. Thus, from general principles, V cannot be completely integrable relative to Q_A, since V is not closed relative to Q_A.

However, we can press the foregoing analysis a bit further. Let $g = (g_0, g_1)$ be any solution to $y' = V(t, y)$ lying entirely in Q_A. As noted earlier, we necessarily have $g_1(t) = t + B$, say. Then, to have $g(t) = (g_0(t), t + B)$ lie entirely in Q_A would require $g_0(t) = (t + B)^2/2 + A$. But $V(t, g(t)) = g'(t)$ requires $d/dt[g_0(t)] = t$. That is, $B = 0$ is forced. In summary, $g_0(t) = t^2/2 + A$ and $g_1(t) = t$ are both forced if g lies entirely in Q_A. Thus, the *only* solution to $y' = V(t, y)$ lying entirely in Q_A is the one from our one-parameter family of lifted solutions above that parametrizes Q_A. Hence, V is indeed 1-suitable relative to Q_A (but far from being completely integrable there).

Note. As indicated in the remark following the corollary to Theorem 1, Example 4 is a counterexample to the notion that V k-suitable with respect to a submanifold Q might imply that $V(N \times Q)$ is contained in ${}_{k+1}M$. For, in the situation at hand, V is 1-suitable relative to Q_A, but, as one readily calculates, $V(t, (s^2/2 + A, s))$ is in ${}_2K$ only if $s = t$. And the circumstance cannot be remedied by shrinking the independent variable domain for t to a smaller open set U in K.

Example 5

Our analytic strategy here is strictly a matter of applying the basic general order theory with no "special" observations. For each $j = 1, \ldots, 5$, we compute $T(\mathbf{i})(T(Q_j))$ and $V(Q_j)$. Then we compare $V(Q_j)$ set-theoretically with both $T(\mathbf{i})(T(Q_j))$ and ${}_3K$, drawing the appropriate conclusions.

Q_1: $T(\mathbf{i})(T(Q_1)) = \{(u, v, v, w; x, y, y, z): w \neq 0\}$ while $V(Q_1) = \{(u, v, v, w; v, 0, 0, 0): w \neq 0\}$. Thus, $V(Q_1) \subseteq T(\mathbf{i})(T(Q_1))$, but $V(Q_1) \cap {}_3K$ is empty. So, V is completely integrable over Q_1, but Q_1 contains no solutions of the form $g = g^{[2]}_{00}$.

Q_2: $T(\mathbf{i})(T(Q_2)) = \{(0, v, v, w; 0, y, y, z): v \neq 0 \neq w\}$, while $V(Q_2) = \{(0, v, v, w; v, 0, 0, 0): v \neq 0 \neq w\}$. Thus, $V(Q_2) \cap T(\mathbf{i})(T(Q_2))$ is empty, which means Q_2 contains no solutions at all (and V is therefore vacuously 2-suitable over Q_2). Note that $V(Q_2) \cap {}_3K$ is also empty.

Q_3: $T(\mathbf{i})(T(Q_3)) = \{(u, v, v, 0; x, y, y, 0): u, v, x, y \text{ in } K\}$, while $V(Q_3) = \{(u, v, v, 0; v, 0, 0, 0): u, v \text{ are in } K\}$. Thus, $V(Q_3)$ is contained in $T(\mathbf{i})(T(Q_3)) \cap {}_3K$, whence V is completely 2-integrable over Q_3. Note also that $Q_3 = (T^2(K))^V$, so that Q_3 contains *all* solutions g of the form $g = g^{[2]}_{00}$.

Q_4: $T(\mathbf{i})(T(Q_4)) = \{(0, v, v, 0; 0, y, y, 0): v \neq 0\}$, while $V(Q_4) = \{(0, v, v, 0; v, 0, 0, 0): v \neq 0\}$. Thus, $V(Q_4) \cap T(\mathbf{i})(T(Q_4))$ is empty, whence Q_4

contains no solutions at all. Again, then, V is vacuously 2-suitable relative to Q_4. In contrast to the situation with Q_2, however, this time we have $V(Q_4) \subseteq {}_3K$.

Q_5: $T(\mathbf{i})(T(Q_5)) = \{(u, v, 0, 0; x, y, 0, 0): v \neq 0\}$, while $V(Q_5) = \{(u, v, 0, 0; v, 0, 0, 0): v \neq 0\}$. Thus, $V(Q_5) \subseteq T(\mathbf{i})(T(Q_5))$, but $V(Q_5) \cap {}_3K$ is empty. In fact $Q_5 \cap {}_2K$ is empty. Thus, V is completely integrable over Q_5, but Q_5 contains no solutions of the form $g = g_{00}^{[2]}$.

Example 6

As in Example 5, we apply the general order theory in its most straightforward aspects, with a final easy computation of solutions at the end.

Q_1: $T(\mathbf{i})(T(Q_1)) = \{(u, v, v, w; x, y, y, z): w \neq 0\}$, while $V(Q_1) = \{(u, v, v, w; v, 0, w, 0): w \neq 0\}$. Thus, $V(Q_1) \cap T(\mathbf{i})(T(Q_1))$ is empty. So, no solutions at all lie in Q_1, whence V is vacuously 2-suitable relative to Q_1, even though $V(Q_1) \cap K_3$ is also empty.

Q_2: $(T(\mathbf{i})(T(Q_2)) = \{(u, v, v, 0; x, y, y, 0): u, v, x, y \text{ in } K\}$, while $V(Q_2) = \{(u, v, v, 0; v, 0, 0, 0): u, v \text{ are in } K\}$. Thus, $V(Q_2) \subseteq T(\mathbf{i})(T(Q_2)) \cap {}_3K$. Hence, V is completely 2-integrable relative to Q_2.

Note that $Q_2 = (T^2(K))^V$, so Q_2 contains *all* solutions g of the form $g = g_{00}^{[2]}$. We can easily compute the generic form for g_{00} here. Namely, $d/dt[g_{01}(t)] \equiv 0$ requires $g_{01}(t) \equiv B \equiv d/dt[g_{00}(t)]$. Thus, the generic g_{00} must have the form $g_{00}(t) = Bt + A$. And one easily verifies that, for such a g_{00}, $g_{00}^{[2]}$ does indeed satisfy $y' = V(y)$.

6. CONCLUSION

The heart of the general order theory presented here is simply Theorem 1 with its various ramifications and uses. Principally, Theorem 1 and its corollary show that, as regards ODEs over M, it makes no sense to even consider vector fields V outside the framework of the spaces ${}_kM$, ${}_kM'$, and ${}_{k+1}M$. For, any solution g to $y' = V(t, y)$ which has the form $g = f^{[k]}$ must lie entirely in ${}_kM$, and $g' = f^{[k+1]}$ must lie entirely in ${}_{k+1}M$. Furthermore, *any* solution g to $y' = V(t, y)$ lying entirely in ${}_kM$ must have g' lying entirely in ${}_kM'$.

Now the spaces ${}_kM$, ${}_kM'$, and ${}_{k+1}M$ are not new as objects of mathematical study or use. They can be regarded as strongly embedded submanifolds in $T^k(M)$, $T^{k+1}(M)$, and $T^{k+1}(M)$, respectively. They can also be regarded as diffeomorphic copies, respectively, of the restricted tangential extensions ${}^kM$, $T({}^kM)$, and ${}^{k+1}M$ introduced by Bowman (1970a). Long before the latter globalization, local versions of ${}^kM$, $T({}^kM)$, and ${}^{k+1}M$ served as the tacit framework for the entire discipline known as extensor analysis. [See Craig (1943, 1964) for a comprehensive treatment of extensor analysis.]

What is new here is the demonstration that the spaces ${}_kM$, ${}_kM'$, and ${}_{k+1}M$ play a central governing role in regard to higher order ODEs over M. Indeed, if one takes into account the diffeomorphisms with the Bowman spaces alluded to above, the full ambient spaces $T^k(M)$ and $T^{k+1}(M)$ are superfluous.

The application of general order studies to higher order dynamical studies can be put in the form of an agenda that is unavoidable *if* initial value problems retain their significance and *if* one requires that solutions to vector field equations be differential lifts of curves from the underlying configuration space M. The author's immediate model for the methodology itself (though certainly not a model for higher order physical law) is the involution/complete lift study—Theorem 3, Proposition 3, and Example 7—carried out in the present paper. To be concrete, let us focus on mechanics. Let us accept as facts (they are) that ${}_kM$ is a strongly embedded submanifold in $T^k(M)$ and that, with $\mathbf{i}$: ${}_kM \to T^k(M)$ denoting the inclusion map, $T(\mathbf{i})(T({}_kM)) = {}_kM'$.

1. Define a regular Lagrangian L: $T({}_kM) \to \mathrm{R}$ as one would over any manifold: there is nothing special in the phrase "higher order." By a well-established procedure one derives the associated Hamiltonian vector field V_L: $T({}_kM) \to T^2({}_kM)$, secure in the knowledge that any solution g to $y' = V_L(y)$ has the form $g = h'$, where h lies entirely in ${}_kM$. But it is *not* generally the case that $(\mathbf{i} \circ h)$ takes the form $(\mathbf{i} \circ h) = f^{[k]}$, where f lies in M. Indeed, the latter circumstance would require, by Theorem 1, that $(\mathbf{i} \circ h)' = T(\mathbf{i}) \circ h' = T(\mathbf{i}) \circ g$ lie entirely in ${}_{k+1}M$. But ${}_{k+1}M$ is a proper subset of ${}_kM = T(\mathbf{i})(T({}_kM))$, whence $T(\mathbf{i})^{-1}({}_{k+1}M)$ is a proper subset of $T({}_kM)$. Simply choose any p in $T({}_kM)$ such that p is not in $T(\mathbf{i})^{-1}({}_{k+1}M)$. With V_L smooth we can, for instance, solve the initial value problem $(V_L, 0, p)$. But the solution g will clearly *not* have the property that $T(\mathbf{i}) \circ g$ lies entirely in ${}_{k+1}M$. Hence, for the associated h lying in ${}_kM$, we cannot have $(\mathbf{i} \circ h) = f^{[k]}$. In summary, for $k > 0$, we are *always* faced with an order problem for V_L, even though we have been careful to start by building the dynamical structure relative to ${}_kM$ rather than relative to all of $T^k(M)$.

2. Rewording parts 3 and 4 of Theorem 2 to fit the immediate circumstance, we have:

Theorem 4. Let L: $T({}_kM) \to \mathrm{R}$ be a regular Lagrangian, with V_L: $T({}_kM) \to T^2({}_kM)$ the associated Hamiltonian vector field. Let $\mathbf{i}$: ${}_kM \to T^k(M)$ denote the inclusion map. Let Q be a strongly embedded submanifold in $T({}_kM)$. Then V_L is completely $(k + 1)$-integrable relative to Q if and only if (a) V_L is closed relative to Q, and (b) π_k and $T(\pi_{k-1})$ agree on $T(\mathbf{i})(Q)$.

Proof. On the one hand, by Theorem 2 part 3, closure [condition (a)] is the criterion for complete integrability. On the other hand, by Theorem 1, part 1, $T(\mathbf{i})$ carries $T({}_kM)$ to ${}_kM'$, whence it also carries the subset Q into ${}_kM'$. Thus, condition (b) really amounts to saying that $T(\mathbf{i})$ carries Q into ${}_{k+1}M$, which is precisely what is needed for $(k + 1)$-suitability in this context.

So step 2 in the agenda comes down to finding one or more Q satisfying the conditions of Theorem 4. Such Q will serve as total subuniverses of motion in regard to $y' = V_L(y)$.

3. Pursue qualitative and asymptotic studies of V_L in relation to Q.

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